A Synthetic Light on the Distributions and their Stochasticity

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Yoshio Kinokuniya*

Abstract

The Present author is compelled to make his efforts to complete Radonian calculus, which has been presented first by him. This paper is dedicated to show a fundamental aspect on the relativity between Radonian operators and the ensembles of functions.

On tne theories of distributions and linear operators, this paper will enable you to find some important facts.

I. Preliminaries ; Convergence Hypothesis

1. Applicability.

To define a function of a set $f(\mathrm{e})$ in the Euclidian space of n dimensions $\mathbf{R}^n$, if the Radonian integration[1]

$$f(\mathrm{e}) = \underset{P \varepsilon e}{\mathfrak{S}} f_P \tag{I;1,1}$$

is used as its formulation, the quantity f_P is the *application* of $f(\mathrm{e})$ at the point P ($\varepsilon\, \mathbf{R}^n$), and $f(\mathrm{e})$ is *applicable* throughout $\mathbf{R}^n$. When $n=1$, it is used to define f_x in the form

$$f_x = f(x+0) - f(x-0) \tag{I;1,2}$$

and if f_x is determined for each x, $f(x)$ may be applicable. The generalization of (I;1,2) is to define f_P as

$$f_P = \lim_{e \to P} f(\mathrm{e}). \tag{I;1,3}$$

To use (I;1,1) is to consider the integral $f(\mathrm{e})$ as to be composed of each quantities f_P, but to use (I;1,3) is to consider the application f_P to be determined from the side of the total quantity $f(\mathrm{e})$. These are the two eyes, which are used in the ınverse directions and are not always promised to give the same aspect ; they conclude in different ways very often.

(1;1,2) means that, the function $f(x)$ can be completely applicable even when it is not determinate at all points. If we use (1;1,1), we can classify all of the points by the two properties : (i) $f_P \geqslant 0$, (ii) $f_P < 0$; but, when we use (1:1,3), this classification may not be so evident. In the latter case, we may use the methode of variation ; i. e. we define two functions $\bar{f}$ and $\underline{f}$ posited as

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* 紀国谷芳雄

1 Cf. Y. Kinokuniya : *A Course of Radonian Calculus* (1953) This booklet will be indicated as *Rad. Cal.*, hereafter. (published by Tanne Trading Co. Ltd. Sapporo, Hokkaido)

$$\bar{f}(\mathrm{e})=\overline{\lim_{e_1\subset e}}f(\mathrm{e}_1),\ \text{and}\ \underline{f}(\mathrm{e})=\underline{\lim_{e_1\subset e}}f(\mathrm{e}_1),$$

then the analogy of the above-stated clssification shall be to classify the sets e by the two properties : (i) $\bar{f}\geqslant 0$, and (ii) $\underline{f}<0$.

If a linear operator $L(\varphi)$ is operatable for any finite real functions $\varphi(P)$ in $\mathbf{R}^n$ (say : $P\in\mathbf{R}^n$), it is convenient to use the functions $\varphi(P;\mathrm{e})$ defined as :

$$\varphi(P;\mathrm{e})=\varphi(P),\ \text{when}\ P\in\mathrm{e}\subset\mathbf{R}^n;$$

and $\qquad \varphi(P;\mathrm{e})=\mathit{empty\ null}^{2}$, when $P\bar{\in}\mathrm{e}$.

Since $\qquad L(\lambda\varphi)=\lambda L(\varphi),$

on positing $\qquad \lambda=1/\varphi(P)$ and $\mathrm{e}=(P)$,

directly we have the left side $(L(\lambda\varphi))$ as equal to τ_P on the formulation

$$L(\varphi)=\mathfrak{S}\tau_P\varphi(P). \tag{I;1,4}$$

This means that the operator $L(\varphi)$ is *Radonian*[3], and when we wiite

$$L(\mathrm{e};\varphi)=\mathfrak{S}\tau_P\varphi(\mathrm{e};P) \tag{I;1,5}$$

the function of a set $L(\mathrm{e};1)$ is applicable. But, when the operator $L(\varphi)$ is not operatable in the ensemble of all the finite real functions $\{\mathrm{F}\}$[4] the supports of which are all compact, the conditions are not so clear.

2. Relativity between the Linear Operators and the Ensmbles of Functions.

When the continuum $[0,1]$ is divided by the points of division (or *partition*)

$$0=\overset{(n)}{x_0}<\overset{(n)}{x_1}<\cdots<\overset{(n)}{x_n}=1$$

and, on denoting the set of $\overset{(n)}{x_k}\ (k=1,2\cdots)$ by $\overset{(n)}{\mathbf{E}}$,

$$\overset{(n)}{\mathbf{E}}\subset\overset{(n+1)}{\mathbf{E}}\quad(n=1,2,\cdots),$$

we can define the set $[0,1]$ as the collection of the points

$$P=\lim\overset{(n)}{x_{k_n}} \tag{I;2,1}$$

and the points

$$\overset{(n)}{x_k} \tag{I;2,2}$$

on condition that

$$\lim_{n}\left\{\operatorname*{Max}_{k}(\overset{(n)}{x_{k+1}}-\overset{(n)}{x_k})\right\}=0.$$

Such is a *process of partition.*[5]

As well as the *point-dimensions*[6], the application of an operator must depend on the partition by which the points of the space are defined to make a continuum. But there is another important relativity on which the system of

2 It means the quantities, which make no other than zero on any summation of any power.

3 It means that $L(\varphi)$ is written in the form (I : 1, 4) τ_P being the application of $L(e,1)=\mathfrak{S}\tau_P$.

4 This symbol will be used to mean the ensemble stated here,

5 Cf. e. g. P. Lévy : *Théorie de l' Addition des Variables Aléatoires*, p. 17 (1937).

6 It is the measure of the point. Cf. Rad. Cal. or Y. Kinokuniya : *On. Continuum.* Mem. Muroran Univ. Eng. 1, No. 3 (1952).

applications depends.

Let the linear operator $L(\varphi)$ be operatable for any function φ of the ensemble $\mathfrak{G}$ (say ; $\varphi \varepsilon \mathfrak{G}$). We shall say that L is *of bounded variation with respect to* $\mathfrak{G}$, when

$$|L(\varphi)|$$

are bounded on condition that $\varphi \varepsilon \mathfrak{G}$ and $|\varphi| \leqslant 1$[7]. In this case, it is directly seen that $L(\varphi_\nu)$ is convergent whenever φ_ν converges uniformly in the ensemble $\mathfrak{G}$. However, it must be noted that the application of the function of a set

$$L(\mathrm{e}; 1)$$

cannot be gained so clearly in case $\mathfrak{G}=\{\mathfrak{C}\}$[8], as in case $\mathfrak{G}=\{\mathfrak{F}\}$. This will be an important relativity in our analysis.

3. Convergence Hypothesis.

It may be evident that we can make an example of a linear operator which is Radonian[9] but $L(\mathrm{e}; \varphi_\nu)$ does not always converge when φ_ν uniformly converges. So it will be significant if we posit here the hypothesis that

$$L(\mathrm{e}; \varphi_\nu) \text{ be convergent}$$

whenever φ_ν uniformly converges. Henceforth this hypothesis will be called the "*convergence hypothesis*". It need hardly be said that this event too, may be related to the ensemble of the functions being considered.

Now, let this linear operator $L(\mathrm{e}; \varphi)$ satisfy the convergence hypothesis, but be not of bounded variation with respect to the ensemble $\mathfrak{G}$; then there can be found the sequence of the functions $\varphi_n \varepsilon \mathfrak{G}$, such that

$$|L(\mathrm{e}; \varphi_n)| \geqslant n^2 \ ; \ |\varphi_n| \leqslant 1.$$

In this case, apparently

$$\varphi_n/n = \psi_n \to 0 \text{ uniformly}$$

so that it must be

$$L(\mathrm{e}; \psi_n) \to 0$$

on account of the convergence hypothesis. This gives an contradiction, since

$$|L(\mathrm{e}; \psi_n)| = |L(\mathrm{e}; \varphi_n/n)| = \frac{1}{n}|L(\mathrm{e}; \varphi_n)| \geqslant n.$$

Thus, it is resulted that :

THEOREM I : *If the linear operator $L(\mathrm{e}; \varphi)$ satisfies the convergence hypothesis with respect to the ensemble $\mathfrak{G}$ to which $c\varphi$ belongs as well as φ, then $L(\mathrm{e}; 1)$ must be of bounded variation with respect to $\mathfrak{G}$.*

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7 This is not a new idea ; a primitive form of this process can be seen, e. g. in N. Bourbaki's "*Integration*" (1952) p. 45, p. 47.

8 The ensemble of the continuous functions the supports of which are compact.

9 On this point, the terminology '*Radonian*' in our calculus is different from that which was introduced by N. Bourbaki, applying the '*measure of Radon*' in his sense. Ouri signification is more general.

II. Derivational Analysis ; Stochasticity

1. Riesz' Relation.

It is known as *Riesz' theorem* that : *if* $L(\mathrm{e};\varphi)$ *is oparatable in* $\{\mathfrak{C}\}$, *and satisfies the convergence hypothesis, then* $L(\mathrm{e};1)$ *is of bounded variation* ; e *denotes the set* $(-\infty, x)$. On this theorem, we shall reach a generalization, if we posit a set e of the n-dimensional Euclidean space $\mathbf{R}^n$ and a general ensemble of functions $\mathfrak{G}$ in places of e and $\{\mathfrak{C}\}$ respectively. Just in the above, in Theorem I, we have stated the consistency of this fact.

It is especialy notable that, this relation is inversible ; i. e. if the function of a set $L(\mathrm{e};1)$ is of bounded variation with respect to $\mathfrak{G}$, then the operator $L(\mathrm{e};\varphi)$ satisfies the convergence hypothesis with respect to $\mathfrak{G}$. The demonstration is easy. Consequently we see :

THEOREM II. *The convergence hypothesis and the property of bounded varivtion are equivalent with respect to the same ensemble of functions.*

Henceforth this will be called *Riesz' relation.*

2. Derivation in Radonian Calculus.

The density of a distribution $L(\mathrm{e};1)$ cannot be generally determinate ; but when $L(\mathrm{e}, 1)=F(x)$ is posited as derivable with respect to

$$\mathrm{e}=(-\infty, x),$$

it can be considered as determinate. The density will be defined as $f(x)$, when we posit it in the form

$$F(x_1)-F(x)=f(x)(x_1-x)+(x_1-x)\varepsilon_1 \tag{II;2,1}$$

and suppose

$$\varepsilon_1\to 0, \text{ as } x_1\to x.$$

The generalization of (II;2,1) may be posited, in Radonian Calculus, as written in the form

$$\psi(x_1)-\psi(x)=\{F(x_1)-F(x)\}\{\varphi(x)+\varepsilon_1\} \tag{II;2,2}$$

on condition

$$\varepsilon_1\to 0, \text{ as } x_1\to x,$$

where

$$\psi(x)=\underset{-\infty}{\overset{x}{\mathfrak{S}}}\tau_u\varphi(u)\ ;\ \varphi(x),\ \psi(x)\in\mathfrak{G}.$$

Moreover, it will be natural to take the ratio

$$\tau_x/\mu_x=f(x) \tag{II;2,3}$$

μ_x being the point-dimension at the point x, though $f(x)$ may not be always determinate. It need hardly be said that $f(x)$ indicates the density of the distribution

$$F(x)=\underset{-\infty}{\overset{x}{\mathfrak{S}}}\tau_u \tag{II;2,4}$$

at the point x.

If $\psi(x)\in\mathfrak{G}$ uniquely corresponds to any $\varphi(x)\in\mathfrak{G}$ and inversely, then the operator

$$L(e;\varphi)=\psi(x)$$

is *inversible*. In this section I will state some important facts on inversibility, from the viewpoint of derivability of ψ and φ in the ordinary sense.

If

$$\underline{\lim}\frac{|F(x+\varepsilon_1)-F(x-\varepsilon_2)|}{\varepsilon_1+\varepsilon_2}\geqslant n$$

when $\varepsilon_1,\ \varepsilon_2\to+0$,

we write

$$|\tau_x|/\mu_x\geqslant n \quad\text{or}\quad |f(x)|\geqslant n. \tag{II;2,5}$$

Now let us denote by e_n the set of the points x at which the inequality (II;2,5) effects, and supose the length of the interval E is equal to $2K$. Then.

$$\sum_{n=1}^{\infty} nm(e_n-e_{n+1})\leqslant\underset{\mathbf{E}}{\mathfrak{S}}\mu_x\leqslant\sum_{n=0}^{\infty}(n+1)m(e_n-e_{n+1})$$

and

$$0<\underset{\mathbf{E}}{\mathfrak{S}}\mu_x=2K<+\infty,$$

so that the series of positve terms

$$\sum_{n=1}^{\infty} nm(e_n-e_{n+1}) \tag{II;2,6}$$

must be convergent. Therefore, for the sufficiently large integers N, we may have

$$0\leqslant m(e_N)<\frac{\delta}{N}$$

on a positive constant δ, since

$$\sum_{n\geqslant1} nm(e_n-e_{n+1})\geqslant Nm(e_N)$$

where the left side tends to zero as $N\to\infty$ on account of (II;2,6). Thus it is concluded that

$$m(e_N)\to0 \quad\text{as}\quad N\to\infty. \tag{II;2,7}$$

Next, let us take the system of applications $\overset{(N)}{\tau_x}$ defined as :

$$\overset{(N)}{\tau_x}=\tau_x\ ,\ \text{when } x\in\mathbf{E}-e_N$$

and

$$\overset{(N)}{\tau_x}=\textit{empty null},\ \text{when } x\,\bar{\in}\,E-e_N,$$

to define the Radonian integrals

$$\overset{(N)}{\psi}(x)=\overset{(N)}{L}(x,\varphi)=\overset{x}{\underset{-\infty}{\mathfrak{S}}}\varphi(u)\overset{(N)}{\tau_u}$$

and the distributions

$$\overset{(N)}{F}(x)=\overset{x}{\underset{-\infty}{\mathfrak{S}}}\overset{(N)}{\tau_u}\ .$$

Then directly we have

$$|\overset{(N)}{F}(x_1)-\overset{(N)}{F}(x)|\leqslant\mathfrak{S}|\overset{(N)}{\tau_u}|\leqslant N|x_1-x|$$

for any pair of x and x_1, so that the relation (II; 2,2) may be written in the form

$$\frac{\overset{(N)}{\psi}(x_1)-\overset{(N)}{\psi}(x)}{x_1-x}=\varphi(x)\frac{\overset{(N)}{F}(x_1)-\overset{(N)}{F}(x)}{x_1-x}+\overset{(N)}{\varepsilon_1}\,;$$

$$\overset{(N)}{\varepsilon_1}\to 0 \text{ as } x_1\to x.$$

This means that derivability of $\overset{(N)}{\psi}(x)$ causes derivability of $\overset{(N)}{F}(x)$ and inversely.

The limit form of the above is

$$\frac{d\overset{(N)}{\psi}(x)}{dx}=\varphi(x)\frac{d\overset{(N)}{F}(x)}{dx}.$$

Therefore, if $\overset{(N)}{\psi}(x)$ and $\varphi(x)$ are n-times continuously derivable, the function $\overset{(N)}{\mathrm{F}}(x)$ must be so too and

$$\frac{d^n\overset{(N)}{\psi}(x)}{dx^n}=\sum_{k=0}^{n-1}\frac{d^k\varphi(x)}{dx^k}\,\frac{d^{n-k}\overset{(N)}{F}(x)}{dx^{n-k}}c_k^{n-1}\,. \tag{II; 2,8}$$

To restrict the functions φ and $\overset{(N)}{\psi}$ within the ensemble of the functions n-times continuously derivable, means not to prohibit us to choose both of the functions as more than n-times derivable. This being so, we easily see that $\overset{(N)}{F}(x)$ must be infinite times continuously derivable; so, evidently it follows that we might not apply both of φ and $\overset{(N)}{\psi}$ as exactly n-times derivable (but not $(n+1)$ times derivable). Consequently we have :

THEOREM III : *If the Radonian operator*

$$L(x;\varphi)=\underset{-\infty}{\overset{x}{\mathfrak{S}}}\varphi(u)\tau_u$$

is inversible with respect to $\{\mathfrak{C}\}$, *there can be found a sequence of Radonian operators*

$$\overset{(N)}{L}(x;\varphi)=\underset{-\infty}{\overset{x}{\mathfrak{S}}}\varphi(u)\overset{(N)}{\tau_u}\;(N=1,2\cdots)$$

which satisfy the conditions

(i) $\overset{(N)}{\tau_x}=\tau_x$, *when* $x\in \mathbf{E}-\overset{(N)}{\mathbf{e}}$,

(ii) $\overset{(N)}{\tau_x}=$ *empty unll, when* $x\,\bar{\in}\,\mathbf{E}-\overset{(N)}{\mathbf{e}}$,

(iii) $m(\overset{(N)}{\mathbf{e}})\to 0$ *as* $N\to\infty$,

and cannot be inversible with respect to the functions finite times continuously derivable the supports of which are all involved in $\mathbf{E}$; *i.e. if* $\overset{(N)}{L}(x,\varphi)=\overset{(N)}{\psi}$ *is inversible with respect to an ensemble* $\mathfrak{G}$, $\mathfrak{G}$ *must be the ensemble of the functions* *infinite times derivable.*

3 Distributional Derivation.

Besides the ensemble $\{\mathfrak{C}\}$, the esemble $\left\{\mathfrak{C}, \int\right\}$ plays an important role, which is the collection of $\varphi \varepsilon \{\mathfrak{C}\}$ and $\Phi = \int_{-\infty}^{x} \varphi(u)du$. Any element of $\{\mathfrak{C}\}$ has its support as compact, whereas the support of $\Phi(x)$ is not generally compact, since

$$\Phi(x) = \Phi(\alpha) \text{ for } x > \alpha$$

when $\varphi(x)=0$ in the same range. Hence, a Radonian operator $L(\varphi)$ cannot be generally operatable in $\left\{\mathfrak{C}, \int\right\}$, though $L(\varphi)$ is posited as operatable in $\{\mathfrak{C}\}$; to be exact, the operatability

$$L(\varphi) = \underset{-\infty}{\overset{\infty}{\mathfrak{S}}} \tau_x \varphi(x), \quad \varphi \varepsilon \{\mathfrak{C}\},$$

does not always cause the operatability of $L(\varphi)$ with respect to $\left\{\mathfrak{C}, \int\right\}$, whereas

$$L(x, \psi) = \underset{-\infty}{\overset{x}{\mathfrak{S}}} \tau_u \psi(u)$$

is operatable in $\left\{\mathfrak{C}, \int\right\}$, whenever $L(x, \varphi)$ is operatable in $\{\mathfrak{C}\}$.

On partial differentiation,

$$f'(x; \varphi) \equiv \int_{-\infty}^{x} f'(t)\varphi(t)dt$$

$$= -\int_{-\infty}^{x} f(t)\varphi'(t)dt + f(x)\varphi(x)$$

when both $f(x)$ and $\varphi(x)$ belong to $\{\mathfrak{C}\}$. So we have

$$f'(\varphi) \equiv \int_{-\infty}^{\infty} f'(x)\varphi(x)dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx = -f(\varphi').$$

The definition of destributional derivation is

$$L'(\varphi) = -L(\varphi')$$

that has been shown for the first time by L. Schwartz[10]. This operation is impertinent to apply in the ensemble $\left\{\mathfrak{C}, \int\right\}$, because then it must take the form

$$f'(\varphi) = -f(\varphi') + f(\infty)\varphi(\infty)$$

and the second term in the right side does not generally vanish in case $\varphi \varepsilon \left\{\mathfrak{C}, \int\right\}$.

With the above-stated preliminary investigations, it will be easily seen that a Radonian operator $L(\varphi)$ operatable in $\left\{\mathfrak{C}, \int\right\}$ must satisfy the condition

10 L. Schwartz , *Théorie des Distributions* I (1950).

$$L(1)=0^{11},$$

when it is distributionally derivable there.

Let us define the *distributional derivation* by the limiting process

$$\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\underset{-\infty}{\overset{x}{\mathfrak{S}}}\varphi(u)(\tau_{u+\varepsilon}-\tau_u)\equiv L'(x;\varphi) \qquad \text{(II; 3, 1)}$$

and suppose $L(\mathfrak{e},1)$ is of bounded vairation with respect to $\{\mathfrak{S}\}$.
We may have generally

$$\underset{-\infty}{\overset{x}{\mathfrak{S}}}\varphi(u)(\tau_{u+\varepsilon}-\tau_u)=\underset{-\infty}{\overset{x}{\mathfrak{S}}}\{\varphi(u)-\varphi(u+\varepsilon)\}\tau_{u+\varepsilon}+\underset{x+0}{\overset{x+\varepsilon}{\mathfrak{S}}}\varphi(u)\tau_u,$$

so that, if $\varphi(u)$ is 2 times continuoasly derivable, it must be

$$L'(x;\varphi)=-L(x;\varphi')+\lim\frac{1}{\varepsilon}\underset{x+0}{\overset{x+\varepsilon}{\mathfrak{S}}}\varphi(u)\tau_u.$$

$$=-L(x;\varphi')+\varphi(x)\lim\frac{\mathfrak{S}\tau_u}{\varepsilon}.$$

When

$$\lim_{\varepsilon\to+0}\frac{L(x+\varepsilon,1)-L(x+0,1)}{\varepsilon}=\mathbf{d}^+L(x,1)$$

exists, it ie the derivative in the right. So, we may state as follows :

THEOREM IV : *For that the Radonian operator $L(x,\varphi)$ be distributionally derivable for any function φ 2 times continuously derivable it is necessary and sufficient that the function $L(x,1)$ be derivable in the right.*

4. Stochasticity.

Let us distinguish the three kinds of derivatives :

(i) *deriavtive in the left* $\mathbf{d}^-F(x)=\lim\{F(x-0)-F(x-\varepsilon)\}/\varepsilon$,

(ii) *deriavtive in the middle* $\tilde{\mathbf{d}}F(x)=\lim\{F(x+\varepsilon)-F(x-\varepsilon)\}/2\varepsilon$,

(iii) *deriavtive in the right* $\mathbf{d}^+E(x)=\lim\{F(x+\varepsilon)-F(x+0)\}/\varepsilon$.

Then we have

$$\tilde{\mathbf{d}}F(x)=\frac{1}{2}\{\mathbf{d}^-F(x)+\mathbf{d}^+F(x)\},$$

since $F(x+0)=F(x-0)$ when $\tilde{\mathbf{d}}$ exists. However, if we take the general definition

$$\overset{*}{\mathbf{d}}F(x)=\lim_{\substack{\varepsilon_1\to0\\ \varepsilon_2\to0}}\frac{F(x+\varepsilon_1)-F(x-\varepsilon_2)}{\varepsilon_1+\varepsilon_2},$$

it will be

$$\overset{*}{\mathbf{d}}F(x)=\lambda\mathbf{d}^-F(x)+\mu\mathbf{d}^+F(x) \qquad \text{(II; 4, 1)}$$

where $\lambda=\lim\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}$ and $\mu=1-\lambda$. Since the parameter λ can take any value of $0\leqslant\lambda\leqslant1$, $\overset{*}{\mathbf{d}}$ cannot be determinate when $\mathbf{d}^-F\neq\mathbf{d}^+F$. Such is a *quasi-Browntan* state.

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11 Cf. *Rad. Cal.*, p. 21.

Now let us suppose, each point of the space $\mathbf{R}^n$ owns one and only one property of the system of n properties $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$, and denote by Ω_k the set of the points which owns the property $\mathfrak{p}_k$ and by $\mu(\mathbf{e})$ the a priori measure[12] of the set $\mathbf{e}$ and

$$\Omega=\Omega_1+\Omega_2+\cdots+\Omega_n,$$
$$\mathbf{e}\cap\Omega_k=\mathbf{e}_k,\ \lambda(\mathbf{e}_k)=\mu(\mathbf{e}_k)/\mu(\mathbf{e}).$$

Then, for any function of a set $F(\mathbf{e})$ we have

$$\frac{F(\mathbf{e})}{\mu(\mathbf{e})}=\frac{F(\mathbf{e}_1)}{\mu(\mathbf{e}_1)}\lambda(\mathbf{e}_1)=\frac{F(\mathbf{e}_2)}{\mu(\mathbf{e}_2)}\lambda(\mathbf{e}_2)+\cdots+\frac{F(\mathbf{e}_n)}{\mu(\mathbf{e}_n)}\lambda(\mathbf{e}_n) \qquad \text{(II; 4,2)}$$

where
$$1=\lambda(\mathbf{e}_1)+\lambda(\mathbf{e}_2)+\cdots+\lambda(\mathbf{e}_n). \qquad \text{(II; 4,3)}$$

when $\mathbf{e}\to P$, the quantity $\lambda(\mathbf{e}_k)$ indicates the asymptotic probability that the point P may own the property $\mathfrak{p}_k$.

In this process, the point P is posited first as the owner of a certain single property $\mathfrak{p}_k$ from $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$; but, by the asymptotic relation (II; 4,3) it is changed to be probable to own each property $\mathfrak{p}_k$ in the probability $\lim \lambda(\mathbf{e}_k)$. Such is a *change of the viewpoint from real-idealism to theoretical phenomenism.* In this kind of change, if it is carried by probabilistic measuring, we may call the process "*stochastic*". At the beginning of our investigation, it is given that each point posesses a single property $\mathfrak{p}_k$, but it is not clear how the asymptotic probability $\lambda(\mathbf{e}_k)$ may emerge, especially wether it may be determinate or not. However, if we assume that

$$\lim_{\mathbf{e}\to P} \lambda(\mathbf{e}_k)=\pi_k \qquad \text{(II; 4,4)}$$

are all determined, the state will be expected to move in conformity with the ratios (II; 4,4), and we may say the state is *stochastically determinate.*

Mathematical Seminar in the Muroran Univ. Eng., Hokkaido.

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12 Cf. *Radical*, p. 1 or the paper *On Continuum* quoted in 6.